1. ГРУПА

[1] $y' - xy = \frac{1}{2}x\sin x\cdot e^{-x^2}\cdot y^3$ – Бернулијева, $z = y^{-2} \Rightarrow z' = -2y^{-3}y'$ ... 15б.

$\dots\ z' + 2xz = -x\sin x\cdot e^{-x^2}$ – линеарна ... 15б.

$\Rightarrow z = e^{-\int 2x\,dx}\left[C - \int x\sin x\cdot e^{-x^2}\cdot e^{\int 2x\,dx}dx\right] = e^{-x^2}\left[C - \int x\sin x\,dx\right] = \frac{1}{e^{x^2}}\cdot(C + x\cos x - \sin x)$, $z = \frac{1}{y^2}$

$\Rightarrow \dots \Rightarrow \underline{y^2 = \frac{e^{x^2}}{C + x\cos x - \sin x}}$, 3б.

[2] $\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda-1)(\lambda^2+1) \Rightarrow \lambda_1 = 1,\ \lambda_{2,3} = \pm i \Rightarrow y_H = C_1e^x + C_2\cos x + C_3\sin x$ 10б.

За $f_1(x) = 4\cos x$, $y_{P_1} = x(A\cos x + B\sin x) \Rightarrow \dots \Rightarrow A = B = -1 \Rightarrow y_{P_1} = -x(\cos x + \sin x)$ 10б.

За $f_2(x) = -8xe^{-x}$, $y_{P_2} = (Cx+D)e^{-x} \Rightarrow \dots \Rightarrow C=2, D=3 \Rightarrow y_{P_2} = (2x+3)e^{-x}$ 10б.

$\Rightarrow y = y_H + y_{P_1} + y_{P_2} = \underline{C_1e^x + C_2\cos x + C_3\sin x - x(\cos x+\sin x) + (2x+3)e^{-x}}$ 3б.

[3] $A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 3 & 1 \\ 2 & -4 & -3 \end{bmatrix} \Rightarrow \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ -1 & 3-\lambda & 1 \\ 2 & -4 & -3-\lambda \end{vmatrix} = (1-\lambda)(\lambda-3)(\lambda+3) + 4(1-\lambda) - 8 + 4(3-\lambda) = (1-\lambda)[\lambda^2-1] = -(\lambda-1)^2(\lambda+1)$

$\det(A-\lambda I) = 0 \Leftrightarrow \underline{\lambda_{1,2} = 1,\ \lambda_3 = -1}$ 10б.

За $\lambda_{1,2} = 1$: $(A-\lambda_{1,2}I)\cdot M_2 = 0$

$\underline{(A-\lambda_{1,2}I)\cdot M_1 = M_2}$

$\begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 1 \\ 2 & -4 & -4 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 1 \\ 2 & -4 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

$\dots \rightarrow M_2 = b_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},\ M_1 = b_1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - b_2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$b_1 = 1, b_2 = 0 \Rightarrow \underline{X_1} = (M_1 + M_2t)e^{\lambda_1 t} = \dots = \underline{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}e^{1\cdot t}}$ 12б.

$b_1 = 0, b_2 = 1 \Rightarrow \underline{X_2} = (M_1 + M_2t)e^{\lambda_2 t} = \dots = \underline{\begin{bmatrix} -2+2t \\ t \\ -1 \end{bmatrix}e^{1\cdot t}}$

За $\lambda_3 = -1$: $\underline{(A-\lambda_3 I)\cdot M = 0}$

$\underline{\begin{bmatrix} 2 & 0 & -2 \\ -1 & 4 & 1 \\ 2 & -4 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$ $\rightarrow \dots \rightarrow M = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \overset{a=1}{=\!=} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{X_3} = Me^{\lambda_3 t} = \underline{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{-1\cdot t}}$ 8б.

$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = C_1X_1 + C_2X_2 + C_3X_3$

$\Rightarrow \underline{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C_1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}e^{t} + C_2\begin{bmatrix} -2+2t \\ t \\ -1 \end{bmatrix}e^{t} + C_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{-t}} \Rightarrow$ $x =$ ~~~, $y =$ ~~~, $z =$ ~~~ 4б.

3. ГРУПА

1 $y' - xy = \frac{1}{2} x\cos x \cdot e^{-x^2} \cdot y^3$ – Бернулијева, $z = y^{-2} \Rightarrow z' = -2y^{-3}y'$ ..... 15б.

.... $z' + 2xz = -x\cos x\, e^{-x^2}$ – линеарна ..... 15б.

$\Rightarrow z = e^{-\int 2x\,dx}\left[C - \int x\cos x \cdot e^{-x^2} \cdot e^{\int 2x\,dx} \cdot dx\right] = e^{-x^2}\left[C - \int x\cos x\,dx\right] = \frac{1}{e^{x^2}}(C - x\sin x - \cos x)$, $z = \frac{1}{y^2}$

$\Rightarrow \dots \Rightarrow \underline{y^2 = \frac{e^{x^2}}{C - x\sin x - \cos x}}$. 3б.

2 $\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda-1)(\lambda^2+1) \Rightarrow \lambda_1 = 1,\ \lambda_{2,3} = \pm i \Rightarrow y_H = C_1e^x + C_2\cos x + C_3\sin x$ 10б.

За $f_1(x) = 8xe^{-x}$, $y_{P_1} = (Ax+B)e^{-x} \Rightarrow \dots \Rightarrow A = -2,\ B = -3 \Rightarrow y_{P_1} = -(2x+3)e^{-x}$ 10б.

За $f_2(x) = -4\cos x$, $y_{P_2} = x(C\cos x + D\sin x) \Rightarrow \dots \Rightarrow C = D = 1 \Rightarrow y_{P_2} = x(\cos x + \sin x)$ 10б.

$\Rightarrow y = y_H + y_{P_1} + y_{P_2} = \underline{C_1e^x + C_2\cos x + C_3\sin x - (2x+3)e^{-x} + x(\cos x + \sin x)}$ 3б.

3 $A = \begin{bmatrix} -3 & -4 & 2 \\ 1 & 3 & -1 \\ -2 & 0 & 1 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -4 & 2 \\ 1 & 3-\lambda & -1 \\ -2 & 0 & 1-\lambda \end{vmatrix} = (\lambda+3)(\lambda-3)(1-\lambda) - 8 + 4(1-\lambda) + 4(3-\lambda) = \dots = -(\lambda-1)^2(\lambda+1)$

$\det(A - \lambda I) = 0 \Leftrightarrow \underline{\lambda_{1,2} = 1,\ \lambda_3 = -1}$ 10б.

За $\lambda_{1,2} = 1$: $(A - \lambda_{1,2}I)\cdot M_2 = 0$

$\underline{(A - \lambda_{1,2}I)\cdot M_1 = M_2}$

$$\left.\begin{array}{l} \begin{bmatrix} -4 & -4 & 2 \\ 1 & 2 & -1 \\ -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -4 & -4 & 2 \\ 1 & 2 & -1 \\ -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \end{array}\right\} \to \dots \to M_2 = b_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},\ M_1 = b_1\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - b_2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$b_1 = 1, b_2 = 0 \Rightarrow \underline{X_1} = (M_1 + M_2t)e^{\lambda_{1,2}t} = \dots = \underline{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} e^{1\cdot t}}$ 12б.

$b_1 = 0, b_2 = 1 \Rightarrow X_2 = (M_1 + M_2t)e^{\lambda_{1,2}t} = \dots = \underline{\begin{bmatrix} -1 \\ t \\ -2+2t \end{bmatrix} e^{1\cdot t}}$

За $\lambda_3 = -1$: $\underline{(A - \lambda_3I)\cdot M = 0}$

$$\left.\begin{bmatrix} -2 & -4 & 2 \\ 1 & 4 & -1 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} \to \dots \to M = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \underset{a=1}{=} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{X_3} = Me^{\lambda_3 t} = \underline{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-1t}}$$ 8б.

$\Rightarrow X = C_1X_1 + C_2X_2 + C_3X_3$

$\Rightarrow \underline{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C_1\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}e^t + C_2\begin{bmatrix} -1 \\ t \\ -2+2t \end{bmatrix}e^t + C_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{-t}} \Rightarrow$ x = ______ y = ______ z = ______ 4б.