2. ГРУПА 16.01.2008.

II КОЛОКВИЈУМ ИЗ МАТЕМАТИКЕ 3

Презиме и име:____, број индекса:_________

1. Решити систем диференцијалних једначина

$$\begin{aligned} x' &= x - 5y \\ y' &= x + 3y + 4\sin 2x \end{aligned}.$$

2. Израчунати $\int\limits_{C^-} \frac{z+1}{e^z+1}dz$, ако је $C = \{ z \mid |z| = 4 \}$.

3. Применом Лапласове трансформације решити једначину $y''' + y' = f(t)$,

ако је $f(t) = \begin{cases} 1 & , \ 0 \le t < 5 \\ 0 & , \ 5 \le t \end{cases}$ и $y(0) = y'(0) = 0$, $y''(0) = 1$.

3. ГРУПА 16.01.2008.

II КОЛОКВИЈУМ ИЗ МАТЕМАТИКЕ 3

Презиме и име:____, број индекса:_________

1. Решити парцијалну диференцијалну једначину

$$y(x^2 - z^2)z'_x - x(z^2 + y^2)z'_y = z(x^2 + y^2).$$

2. Израчунати $\int\limits_{C^+} \frac{9e^z}{z^2(z^2+9)}dz$, ако је $C = \{ z \mid |z + 2i| = 3 \}$.

3. Применом Лапласове трансформације решити систем једначина

$$\begin{aligned} x'' + x' + y'' - y &= 4e^t \\ x' + 2x - y' + y &= 4e^{-t} \end{aligned},$$

ако је $x(0) = y(0) = y'(0) = 0$, $x'(0) = 1$.

2. ГРУПА – II колоквијум '08

1 $\left.\begin{array}{l} x' = x - 5y \\ y' = x + 3y + 4\sin 2t \end{array}\right\} \Rightarrow \det(A-\lambda E) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = \ldots = \lambda^2 - 4\lambda + 8 = 0 \Rightarrow \underline{\lambda_{1,2} = 2 \pm 2i}$

$\lambda_1 = 2+2i: \ (A-\lambda_1 E)\cdot M = \begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \ldots \Rightarrow M = b\begin{bmatrix} 2i-1 \\ 1 \end{bmatrix}$

$\Rightarrow X_{kom} = \begin{bmatrix} 2i-1 \\ 1 \end{bmatrix} e^{(2+2i)t} = \begin{bmatrix} 2i-1 \\ 1 \end{bmatrix} (\cos 2t + i\sin 2t)e^{2t} = \ldots = \underbrace{\begin{bmatrix} -(\cos 2x + 2\sin 2x) \\ \cos 2x \end{bmatrix} e^{2x}}_{X_1} + i \cdot \underbrace{\begin{bmatrix} 2\cos 2x - \sin 2x \\ \sin 2x \end{bmatrix} e^{2x}}_{X_2}$

$\Rightarrow \underline{X_H = C_1 X_1 + C_2 X_2 = C_1 \begin{bmatrix} -(\cos 2x + 2\sin 2x) \\ \cos 2x \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} 2\cos 2x - \sin 2x \\ \sin 2x \end{bmatrix} e^{2x}}$ 16б.

$X_p = \begin{bmatrix} a_1 \cos 2x + a_2 \sin 2x \\ b_1 \cos 2x + b_2 \sin 2x \end{bmatrix}, \ X_p' = \begin{bmatrix} -2a_1 \sin 2x + 2a_2 \cos 2x \\ -2b_1 \sin 2x + 2b_2 \cos 2x \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} -2a_1 \sin 2x + 2a_2 \cos 2x = (a_1 \cos 2x + a_2 \sin 2x) - 5(b_1 \cos 2x + b_2 \sin 2x) \\ -2b_1 \sin 2x + 2b_2 \cos 2x = (a_1 \cos 2x + a_2 \sin 2x) + 3(b_1 \cos 2x + b_2 \sin 2x) + 4\sin 2x \end{array}\right\}$

$\Rightarrow \ldots \Rightarrow \left.\begin{array}{ll} -2a_1 = a_2 - 5b_2 & -2b_1 = a_2 + 3b_2 + 4 \\ 2a_2 = a_1 - 5b_1, & 2b_2 = a_1 + 3b_1 \end{array}\right\} \Rightarrow \ldots \Rightarrow \left.\begin{array}{l} a_1 = -2,\ a_2 = -1 \\ b_1 = 0,\ b_2 = -1 \end{array}\right\}$

$\Rightarrow \underline{X_p = \begin{bmatrix} -2\cos 2x - \sin 2x \\ -\sin 2x \end{bmatrix}}$ 16б. $\Rightarrow \underline{X = \begin{bmatrix} x \\ y \end{bmatrix} = X_H + X_p} = \begin{bmatrix} -C_1(\cos 2x + 2\sin 2x)e^{2x} + C_2(2\cos 2x - \sin 2x)e^{2x} - 2\cos 2x - \sin 2x \\ C_1 \cdot \cos 2x\, e^{2x} + C_2 \cdot \sin 2x\, e^{2x} - \sin 2x \end{bmatrix}$ 4б.

2 $e^z + 1 = 0 \Leftrightarrow e^z = -1 \Rightarrow z = \mathrm{Ln}(-1) = \ln|-1| + i\,\mathrm{Arg}(-1) = 0 + i(\pi + 2k\pi) \Rightarrow \underline{z_1 = i\pi,\ z_2 = -i\pi \in D}$ 10б.

$\mathrm{Rez}(f(z), z_1 = i\pi) = \lim_{z\to i\pi} (z - i\pi)\frac{z+1}{e^z+1} = \lim_{z\to i\pi} \frac{z^2 + (1-\pi i)z - \pi i}{e^z + 1} \overset{\text{Л.П.}}{=} \lim_{z\to i\pi} \frac{2z + (1-\pi i)}{e^z} = -(1+\pi i) \neq 0$

$\mathrm{Rez}(f(z), z_2 = -i\pi) = \lim_{z\to -i\pi} (z + i\pi)\frac{z+1}{e^z+1} = \lim_{z\to -i\pi} \frac{z^2 + (1+\pi i)z + \pi i}{e^z + 1} \overset{\text{Л.П.}}{=} \lim_{z\to -i\pi} \frac{2z + (1+\pi i)}{e^z} = -1 + \pi i \neq 0$

полови 1. реда! 20б.

$\Rightarrow \oint_{C^-} f(z)\,dz = -2\pi i\left[\mathrm{Rez}(f(z), z_1) + \mathrm{Rez}(f(z), z_2)\right] = -2\pi i\left[-1 - \cancel{\pi i} - 1 + \cancel{\pi i}\right] = \underline{4\pi i}$ 3б.

3 $\mathcal{L}[y''' + y'] = \mathcal{L}[f(t)],\ f(t) = u(t-0) - u(t-5)$

$\Rightarrow [s^3 \cdot Y(s) - s^2 \cdot 0 - s^1 \cdot 0 - 1] + [s^1 \cdot Y(s) - 0] = \frac{1}{s} - e^{-5s}\cdot\frac{1}{s}$

$\Rightarrow (s^3 + s)\cdot Y(s) = 1 + \frac{1}{s} - e^{-5s}\cdot\frac{1}{s} \Rightarrow \underline{Y(s) = \frac{1}{s(s^2+1)} + \frac{1}{s^2(s^2+1)} - e^{-5s}\frac{1}{s^2(s^2+1)}}$ 16б.

$\Rightarrow \ldots \Rightarrow Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - e^{-5s}\cdot\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$

$\Rightarrow y(t) = (1 - \cos t) + (t - \sin t) - u(t-5)\cdot[(t-5) - \sin(t-5)]$

$\Rightarrow \underline{y(t) = t + 1 - \sin t - \cos t - u(t-5)\cdot[t - 5 - \sin(t-5)]}$ 18б.

3. ГРУПА – II колоквијум '08

1 $y(x^2-z^2)\cdot z'_x - x(z^2+y^2)\cdot z'_y = z(x^2+y^2) \Rightarrow \frac{dx}{y(x^2-z^2)} = \frac{dy}{-x(z^2+y^2)} = \frac{dz}{z(x^2+y^2)}$ придр. систем

I: $\frac{y\,dx+x\,dy}{y^2(x^2-z^2)-x^2(z^2+y^2)} = \frac{d(xy)}{-z^2(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \Rightarrow d(xy) = -z\,dz \Rightarrow \dots \Rightarrow xy+\frac{z^2}{2}=C_1$. п.и. 14б.

II: $\frac{x\,dx-y\,dy}{xy(x^2-z^2)+xy(z^2+y^2)} = \frac{\frac12 d(x^2-y^2)}{xy(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \Rightarrow d(x^2-y^2) = \frac{2xy}{z}dz = \frac{2C_1-z^2}{z}dz \Rightarrow$

$\Rightarrow \int d(x^2-y^2) = \int 2C_1\frac{1}{z}dz - \int z\,dz \Rightarrow x^2-y^2 = 2C_1\ln|z| - \frac{z^2}{2}+C_2 \Rightarrow x^2-y^2+\frac{z^2}{2}-(2xy+z^2)\ln|z| = C_2$. п.и. 16б.

$\Rightarrow z(x,y):\ F\left(xy+\frac{z^2}{2},\ x^2-y^2+\frac{z^2}{2}-(2xy+z^2)\ln|z|\right)=0$ 3б., $F(\cdot,\cdot)$ пр. диф. ф-ја

2 $f(z)=\frac{9e^z}{z^2(z^2+9)} \Rightarrow z_1=0$ пол 2. реда, $z_2=-3i$ пол 1. реда, $z_3=3i\notin D$ 4б.

$\mathrm{Rez}(f(z), z_1=0) = \lim_{z\to0}[z^2 f(z)]' = \lim_{z\to0}\left(\frac{9e^z}{z^2+9}\right)' = \lim_{z\to0}\frac{9e^z(z^2+9)-2z\cdot9e^z}{(z^2+9)^2} = \frac{9\cdot1\cdot9-0}{9^2} = 1$ 13б.

$\mathrm{Rez}(f(z), z_2=-3i) = \lim_{z\to-3i}(z+3i)f(z) = \lim_{z\to-3i}\frac{9e^z}{z^2(z-3i)} = \frac{9e^{-3i}}{-9(-6i)} = \frac{e^{-3i}}{6i} = -\frac{i}{6}e^{-3i}$. 13б.

$\Rightarrow \oint_{C^+} f(z)\,dz = 2\pi i[\mathrm{Rez}(f(z),z_1)+\mathrm{Rez}(f(z),z_2)] = 2\pi i\left[1-\frac{i}{6}e^{-3i}\right] = \pi\left[\frac13 e^{-3i}+2i\right]$. 3б.

3 $\left.\begin{aligned}\mathcal{L}[x''+x'+y''-y] &= 4\mathcal{L}[e^t]\\ \mathcal{L}[x'+2x-y'+y] &= 4\mathcal{L}[e^{-t}]\end{aligned}\right\} \Rightarrow \left.\begin{aligned} s^2X(s)-1+s^1X(s)+s^2Y(s)-Y(s) &= 4\cdot\frac{1}{s-1}\\ s^1X(s)+2X(s)-s^1Y(s)+Y(s) &= 4\cdot\frac{1}{s+1}\end{aligned}\right\}\Rightarrow$

$\Rightarrow \left.\begin{aligned}(s^2+s)\cdot X(s)+(s^2-1)\cdot Y(s) &= 1+\frac{4}{s-1}\\ (s+2)\cdot X(s)+(1-s)\cdot Y(s) &= \frac{4}{s+1}\end{aligned}\right\},\ D=\begin{vmatrix}(s^2+s) & (s^2-1)\\ (s+2) & (1-s)\end{vmatrix} = -2(s-1)(s+1)^2$

$D_X=\begin{vmatrix}\left(1+\frac{4}{s-1}\right) & (s^2-1)\\ \frac{4}{s+1} & (1-s)\end{vmatrix} = -(5s-1),\ D_Y=\begin{vmatrix}(s^2+s) & \left(1+\frac{4}{s-1}\right)\\ (s+2) & \frac{4}{s+1}\end{vmatrix} = 3s-6-\frac{12}{s-1}$

$\Rightarrow X(s)=\frac{D_X}{D} = \frac{5s-1}{2(s-1)(s+1)^2} = \dots = \frac{1/2}{s-1}-\frac{1/2}{s+1}+\frac{3/2}{(s+1)^2}$ 16б. 6б.

$Y(s)=\frac{D_Y}{D} = \frac{3s-6}{-2(s-1)(s+1)^2}+\frac{\cancel{12}\,6}{\cancel{2}(s-1)^2(s+1)^2} = \dots = \frac{9/8}{s+1}-\frac{9/8}{s-1}-\frac{3/4}{(s+1)^2}+\frac{3/2}{(s-1)^2}$

$\Rightarrow x(t) = \frac12 e^t-\frac12 e^{-t}+\frac32 te^{-t} = \mathrm{sh}\,t+\frac32 te^{-t}$ 12б.

$y(t) = \frac98 e^{-t}-\frac98 e^{t}-\frac34 te^{-t}+\frac32 te^{t} = -\frac94\mathrm{sh}\,t-\frac34 te^{-t}+\frac32 te^{t}$